Föreläsning 6/11-13

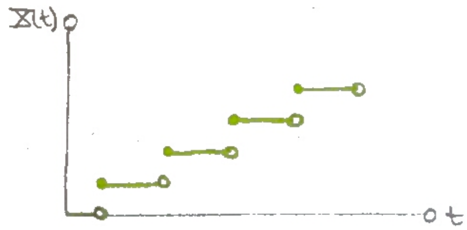
Random processes with stationary and independent increments, In continous time $(X(t), t \geq 0)$

$X(t) - X(s)$ is independent of $(X(r), r \leq s)$ for $s \leq t$
(independent increments)

$X(t+h) - X(s+h)$ has a distribution independent h, only depends on $t-s$.
$(h = -s \Rightarrow X(t-s) - X(0))$
(stationary increments)

Thm For stationary independent increment process with $X(0)=0$ we have $\mu_X(t) = E(X(t)) = \zeta t = E(X(1))t$
$Var(X(t)) = Var(X(1))t$
$K_X(s,t) = Cov(X(s), X(t)) = Var(X(1)) \min(s,t)$.

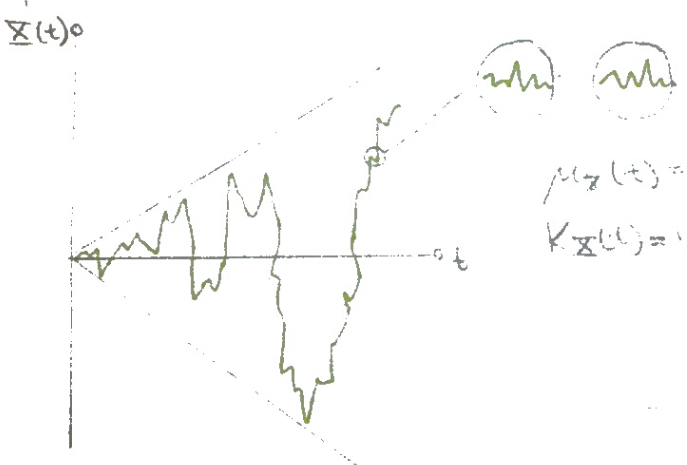
example
Poisson process with intensity $\lambda > 0$, $(X(t), t \geq 0)$, stationary independent increment process with $X(t) - X(s)$ $Po(\lambda(t-s))$-distributed for $s \leq t$. $X(0) = 0$



$\mu_X(t) = E(X(1))t = E(Po(\lambda))t = \lambda t$

$K_X(t) = Var(X(1)) \min(s,t) = Var(Po(\lambda)) \min(s,t)$
$= \lambda \min(s,t)$

example
Wiener process = Browniar motion (most important of all random processes), $(X(t), t \geq 0)$, stationary independent increment process $X(t) - X(s)$ $N(0, \sigma^2(t-s))$-distributed, $X(0) = 0$



doesn't care about the history, looks the same though we "zoom in".

$\mu_X(t) = E(X(1))t = 0$

$K_X(t) = Var(X(1)) \min(s,t) = \sigma^2 \min(s,t)$.

Thm Wiener process is Gaussian process (normal process)

Proof $\sum_{i=1}^{n} a_i X(t_i) = \{\text{say } 0 \le t_1 \le t_2 \le \ldots \le t_n\} =$

$= a_n (X(t_n) - X(t_{n-1})) + (a_n + a_{n-1})(X(t_{n-1}) - X(t_{n-2})) +$

$+ (a_n + a_{n-1} + a_{n-2})(X(t_{n-2}) - X(t_{n-3})) + \ldots + (a_n + \ldots + a_2)(X(t_2) - X(t_1)) +$

$+ (a_n + \ldots + a_1) X(t_1)$ which is normaldistributed because of independent normaldistributed increments. ▨

Consequence: We can alternatively define the Wiener process as a normal process with $\mu_X = 0$ and $K_X(s,t) = R_X(s,t) = \sigma^2 \min(s,t)$

Random Walk

$X_n = \sum_{i=1} Y_i$, $n = 0, 1, 2, \ldots$ where $Y_1, Y_2, \ldots$ are IID (independent identically distr.) r.v's.

Discrete time version of stationary independent increment process.

Many (most (all?)) calculations for stationary independent increment processes use stationary independent increments.

example

$P(X(1) \le 1, X(2) \le 2)$ for Poisson-process?

$= P(X(1) = 0, X(2) - X(1) \le 2) + P(X(1) = 1, X(2) - X(1) \le 1) = \{\text{independent incr.}\} =$

$= \underbrace{P(X(1)=0)}_{e^{-\lambda}} \underbrace{P(X(1) \le 2)}_{(1+\lambda+\frac{\lambda^2}{2})e^{-\lambda}} + \underbrace{P(X(1)=1)}_{\lambda e^{-\lambda}} \underbrace{P(X(1) \le 1)}_{(1+\lambda)e^{-\lambda}}$

example

$P(X(1) \le 1, X(2) \le 2)$ for Wiener-process?

$= P(X(1) \le 1, X(2) + X(1) - X(1) \le 2) = \int_{-\infty}^{\infty} P(X(1) \le 1, X(2) - X(1) + X(1) \le 2 \mid X(1) = x) \, f_{X(1)}(x)dx$

$= \int_{-\infty}^{1} P(\underbrace{X(2) - X(1)}_{N(0,\sigma^2)} \le 2 - x) \underbrace{f_{X(1)}(x)}_{N(0,\sigma^2)} dx = \int_{-\infty}^{1} \Phi\left(\frac{2-x}{\sigma}\right) \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} dx$

Section 5.8 Martingales

Martingales are random processes based on the idea of "fair game". As a preparation for this we have to talk a bit more about conditional expectations...

$E(Y \mid X_1 = x_1, \ldots, X_n = x_n) = \int_{-\infty}^{\infty} y \underbrace{\frac{f_{Y, X_1, \ldots, X_n}(y, x_1, \ldots, x_n)}{f_{X_1, \ldots, X_n}(x_1, \ldots, x_n)}}_{f_{Y \mid X_1, \ldots, X_n}(y \mid x_1, \ldots, x_n)} dy = g(x_1, \ldots, x_n)$

$$E(Y|X_1,\dots,X_n) = g(\underbrace{X_1,\dots,X_n}) = E(Y|F_n)$$

F_n information about $X_1,\dots,X_n$

$\hookrightarrow$ = σ-field based on the

Definition $(X_n)_{n=0}^{\infty} = (X_n, n \geq 1)$ is martingale if $\underbrace{E(X_n|F_m)}_{X_1,\dots,X_m} = X_m$, $m \leq n$

example

$X_n = \sum_{i=1}^{n} Y_i$ with Y_i IID is martingale if $E(Y_i) = 0$.